TWMS J. Pure Appl. Math., V.17, N.1, 2026, pp.106-123
DOI:10.30546/2219-1259.17.1.2026.106

NECESSARY AND SUFFICIENT CONDITIONS FOR ABSOLUTE MONOTONICITY RELATED TO THE GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND

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Abstract. This paper focuses on the generalized elliptic integral of the first kind and a related function constructed by combining this integral with a logarithmic asymptotic function. We establish the necessary and sufficient conditions for the parameter in the logarithmic term to ensure that the arbitrary-order derivative of the constructed function is absolutely monotonic on the interval $(0, 1)$. This result solves two previously proposed problems, providing a negative answer to one and a stronger conclusion to the other. Notably, the method employed in this study can be extended to investigate the absolute monotonicity of zero-balanced hypergeometric functions. As applications, several new functional inequalities involving the generalized elliptic integral of the first kind are derived.

Keywords: generalized elliptic integral, asymptotic function, absolute monotonicity, Jurkat's criterion, necessary and sufficient condition.

AMS Subject Classification: 33C75, 26D07.

1. Introduction

Given real numbers a, b and c with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined on $(0, 1)$ as:

$$F(a, b; c; x) \equiv {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where $(a)_n = \Gamma(n + a)/\Gamma(a)$ (the shifted factorial function or Pochhammer symbol) for $n \in \mathbb{N}$, and $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ (with $\mathrm{Re}(x) > 0$) is the Gamma function [3]. When $c = a + b$, the resulting function $F(a, b; a + b; x)$ is said to be *zero-balanced*. In particular, $F(a, b; a + b; x)$ has a logarithmic singularity at $x = 1$ and satisfies the Ramanujan asymptotic formula as $x \to 1^-$,

$$F(a, b; a + b; x) = \frac{R(a, b) - \ln(1 - x)}{B(a, b)} + O((1 - x) \ln(1 - x)), \quad (1)$$

which can be found in the literature [3, 1.48, p.18], where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the classical Beta function and

$$R(a, b) = -2\gamma - \psi(a) - \psi(b), \quad R(1/2, 1/2) = 4 \ln 2$$

is the Ramanujan constant, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the Psi function and γ is the Euler-Mascheroni constant.

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Manuscript received May 2025.

One of the most important special cases of $F(a,b;a+b;x)$ is the Legendre's complete elliptic integral of the first kind, which is given, for $r \in (0,1)$, such that

$$\mathcal{K}(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{-1/2} d\theta = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right).$$

This integral has been expressed in two forms of single-parameter generalized elliptic integrals; one form, for $a \in (0, 1/2]$, is given by:

$$\mathcal{K}_a(r) = \frac{\pi}{2\sin(\pi a)} F(a, 1-a; 1; r^2) \tag{2}$$

(refer to [9, (1.6)] for the more general case) and the other defined by

$$K_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2)$$

was first introduced in [4]. Clearly, these two definitions differ only slightly, differing by a constant factor. At this time, by (1), the asymptotic formula for $\mathcal{K}_a$ can be presented as:

$$\mathcal{K}_a(r) \sim \ln\left(e^{R(a)/2}/r'\right), \tag{3}$$

where and in what follows $R(a) = R(a, 1-a)$ and $r' = \sqrt{1-r^2}$, while it will be more complicated for $K_a(r)$. For the special case $a = 1/2$, the asymptotic function for $\mathcal{K}$ reduces to $\ln(4/r')$. By this, there are many known properties of various functions related to $\mathcal{K}(r)$ and $\ln(4/r')$, including monotonicity, convexity, concavity, absolute monotonicity and inequalities (cf. [2, 8, 14, 15, 16, 18, 19, 20]), several of which have been extended to generalized elliptic integral of the first kind [6, 7, 13, 17, 22, 23]. If we consider the ratio of $\mathcal{K}$ and $\ln(4/r')$, choosing either of the above definitions may not seem to be affected, but it will be very different if we consider their difference. It is precisely because of this that we adopt the definition of (2) in this paper.

In a certain sense, there exists a better asymptotic function $\ln(1+4/r')$ for $\mathcal{K}$, which is closer to $\mathcal{K}$ than $\ln(4/r')$ at $x = 0$. Since the inequality

$$\mathcal{K}(r) < \ln\left(1 + \frac{4}{r'}\right) - \left(\ln 5 - \frac{\pi}{2}\right)(1-r)$$

for all $r \in (0,1)$ was established [11], some researchers began to approximate the complete elliptic integral of the first kind by $\ln(1+4/r')$. For instance, the authors of the paper [19] proved the convexity of $F(x) = \mathcal{K}(\sqrt{x}) - \ln\left(1 + 4/\sqrt{1-x}\right)$, which had been extended to the generalized complete elliptic integral of the first kind in [6]. At the end of [19], they also posed a conjecture that $F''(x)$ is absolutely monotonic on $(0,1)$, which was solved in [16, Theorem 1.9]. A perfectly natural idea is to study the absolute monotonicity of various derivative of $\mathcal{F}_c(x)$, and further extend it to the zero-balanced hypergeometric function, where

$$\mathcal{F}_c(x) = \mathcal{K}_a(\sqrt{x}) - \ln\left(1 + \frac{c}{\sqrt{1-x}}\right). \tag{4}$$

This idea has recently been achieved by the authors of [5], in which the authors obtained the absolute monotonicity for the first few derivative of F_c (defined in Corollary 1.1), explicitly,

- if $c \in [\sqrt{3}, \pi/(4-\pi)]$, then F_c' is absolutely monotonic on $(0,1)$;
- if $c \in [\sqrt{3}, \kappa_1]$, then F_c'' is absolutely monotonic on $(0,1)$, where $\kappa_1 = \frac{4(2+3\sqrt{4-\pi})}{32-9\pi} - 1$ is the unique positive root of $(9\pi - 32)c^2 + 2(9\pi - 24)c + 9\pi = 0$;
- if $c \in [\sqrt{3}, \kappa_0]$, then F_c''' is absolutely monotonic on $(0,1)$, where $\kappa_0 \approx 4.62357$ is the unique positive root of $(75\pi-256)c^5+(150\pi-416)c^4+192c^3+(480-150\pi)c^2-75\pi c-256 = 0$.

They also studied the absolute monotonicity of $-\left[K_a(\sqrt{x})-\ln(1+c/\sqrt{1-x})\right]'$ with $a\in(0,1/2)$ by using the similar method. In [5], it is to be regretted that the authors did not find necessary and sufficient conditions for the absolute monotonicity, mainly because the method they used is to estimate the sign of Maclaurin's coefficients u_n of $\mathcal{F}_c'$ by scaling and reducing it, which means finding a sequence slightly smaller than u_n to determine its sign. This approach inherently narrows the valid range of c, for example, they required $c\geq\sqrt{3}$. In other words, we will never be able to find the optimal range of c such that $\mathcal{F}_c'$, $\mathcal{F}_c''$ and $\mathcal{F}_c'''$ are absolutely monotonic on $(0,1)$. Even if we use this method to prove the absolute monotonicity of the higher-order derivative function $\mathcal{F}_c^{(k)}$ $(k\geq 4)$, it will be a very tedious and complicate task, and cannot also be expressed by a unified form. It is because of this, they raised two problems to improve their results at the end of [5], namely finding the necessary and sufficient conditions such that $\mathcal{F}_c'$, $\mathcal{F}_c''$ are $\mathcal{F}_c'''$ are absolutely monotonic on $(0,1)$. Fortunately, to address the gap in [5]–where only partial results on the absolute monotonicity of $\mathcal{F}_c^{(k)}(x)$ were obtained without necessary and sufficient conditions–we propose a novel four-step approach. This method not only resolves the open problems in [5] but also extends the conclusions to higher-order derivatives and zero-balanced hypergeometric functions. The key steps are as follows:

Step 1: Represent $\mathcal{F}_c'(x)$ by a power series with coefficient relationship

First, we define an auxiliary function $\mathcal{G}_c(x)$ to decompose the first derivative of $\mathcal{F}_c(x)$ (defined in (4)):

$$\begin{aligned}\mathcal{G}_c(x)&=\frac{a(1-a)\pi}{2\sin(\pi a)}F\left(a,1-a;2;x\right)-\frac{c}{2(c+\sqrt{1-x})}\\&=\sum_{n=0}^{\infty}A_nx^n-\sum_{n=0}^{\infty}B_nx^n=\sum_{n=0}^{\infty}\Big(A_n-B_n\Big)x^n=\sum_{n=0}^{\infty}v_nx^n,\end{aligned}\tag{5}$$

where

- $A_n=\frac{a(1-a)\pi}{2\sin(\pi a)}\frac{\mathcal{W}_n^2}{n+1}$ with $\mathcal{W}_n(a)=\frac{\sqrt{(a)_n(1-a)_n}}{n!}$ (generalized Wallis ratio, reducing to the classic Wallis ratio $W_n=\frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}$ when $a=1/2$);
- B_n $(c\neq 1)$ satisfies the recurrence relation

$$B_n=-\frac{B_{n-1}}{c^2-1}+\frac{cW_n}{2(c^2-1)(2n-1)}\qquad(\text{for } n\geq 1,\ c\neq 1),$$

from the power series expansion of $\frac{c}{2(c+\sqrt{1-x})}$.

Using the derivative property of $\mathcal{F}_c(x)$, we further relate $\mathcal{F}_c'(x)$ to $\mathcal{G}_c(x)$:

$$\mathcal{F}_c'(x)=\frac{1}{1-x}\mathcal{G}_c(x)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}v_k\right)x^n=\sum_{n=0}^{\infty}u_nx^n,\tag{6}$$

where $u_n=\sum_{k=0}^{n}v_k$ (partial sum of v_k). This step converts the problem of analyzing $\mathcal{F}_c^{(k+1)}(x)$ absolute monotonicity into analyzing the non-negativity of u_n. We hope the readers to keep in mind that the parameters a and c in the sequences A_n, B_n, v_n and u_n will be missed if no risk of confusion, unless they are considered as functions.

Step 2: Characterize the monotonicity of u_n

For $c\in[3,\infty)$, we analyze the sign of v_n (and thus the monotonicity of u_n, since $u_n-u_{n-1}=v_n$). Prove that either $v_n\leq 0$ for $n\geq 3$ or there exists $n_0\geq 3$ such that $v_n>0$ for $3\leq n\leq n_0$ and $v_n\leq 0$ for $n\geq n_0+1$. Equivalently, u_n is increasing for $2\leq n\leq n_0$ and is decreasing for $n\geq n_0$. This "unimodal or decreasing" behavior of u_n is critical: it means the non-negativity of u_n (for all n) only needs to be verified up to a finite threshold n_0. (c.f. Lemma 2.3)

Step 3: Determine the unique positive zero of $u_n(c)$

We further analyze how u_n depends on c, proving three key properties:

1. *Monotonicity and root existence.* For any fixed $n \geq 0$, $u_n(c)$ is decreasing in c on $(0, \infty)$, with $u_n(0) > 0$ and $u_n(\infty) < 0$ for any $n \geq 0$. By the Intermediate Value Theorem, $u_n(c)$ has a unique positive zero point, denoted c_n. (c.f. Lemma 2.4)
2. *Lower Bound of c_n.* For all $n \geq 0$, $c_n > 3$. This follows from verifying $u_k(3) > 0$ for $k = 0, 1, 2$) and all $a \in (0, 1/2]$, combined with the conclusion of Step 2. (c.f. Lemma 2.5).
3. *Order of c_n.* The roots satisfy $c_0 < c_1 < c_2$. (c.f. Lemma 2.5)

By Steps 2 and 3, the (unimodal) monotonicity of $\{u_n\}_{n\geq 2}$ shows that $\mathcal{F}_c^{(k+1)}$ $(k \geq 0)$ is absolutely monotonic on $(0, 1)$ if $c \in [3, c_k]$.

Step 4: Derive the necessary and sufficient condition

Combining Steps 1-3, we extend the valid range of c (for $\mathcal{F}_c^{(k+1)}(x)$ to be absolutely monotonic) from $[3, c_k]$ (Step 3) to $(0, c_k]$ by the following expression

$$\mathcal{F}_c'(x) = \mathcal{F}_3'(x) + \frac{3 - c}{2(3 + \sqrt{1 - x})(c + \sqrt{1 - x})\sqrt{1 - x}},$$

which is contained in the following theorem.

Theorem 1.1. *Let $c \in (0, \infty)$ and $\mathcal{F}_c$ be defined on $(0, 1)$ by (4). Then, for any integer $k \geq 0$, the function $\mathcal{F}_c^{(k+1)}(x)$ (the $(k + 1)$-th derivative of $\mathcal{F}_c(x)$) is absolutely monotonic on $(0, 1)$ if and only if $c \in (0, c_k]$. Here, c_k denotes the unique positive zero of $u_k(c)$, where $u_k(c)$ is the k-th coefficient in Maclaurin series expansion of $\mathcal{F}_c'(x)$ as defined in (6).*

Taking $k = 0, 1, 2$ into Theorem 1.1, we obtain the following theorem.

Theorem 1.2. *Let $c \in (0, \infty)$ and $\mathcal{F}_c$ be defined as in Theorem 1.1. For $k = 0, 1, 2$, the necessary and sufficient conditions for absolute monotonicity are explicitly given as follows:*

(1) *$\mathcal{F}_c'(x)$ is absolutely monotonic on $(0, 1)$ if and only if $c \in \left(0, \frac{a(1-a)\pi}{\sin(\pi a) - a(1-a)\pi}\right]$;*
(2) *$\mathcal{F}_c''(x)$ is absolutely monotonic on $(0, 1)$ if and only if $c \in (0, c_1]$, where c_1 is the positive root of the quadratic-type expression:*

$$c_1 = \frac{2a(1 - a^2)(2 - a)\pi - 3\sin(\pi a) + \sqrt{9\sin^2(\pi a) - 4a(1 - a^2)(2 - a)\pi\sin(\pi a)}}{4\sin(\pi a) - 2a(1 - a^2)(2 - a)\pi};$$

(3) *$\mathcal{F}_c'''(x)$ is absolutely monotonic on $(0, 1)$ if and only if $c \in (0, c_2]$, where c_2 is the unique positive root of the equation:*

$$2a(1 - a^2)(3 - a)(4 - a^2)\pi(1 + c)^3 = 3c(15 + 21c + 8c^2)\sin(\pi a).$$

When $a = 1/2$, the generalized elliptic integral $\mathcal{K}_a(r)$ reduces to the complete elliptic integral of the first kind $\mathcal{K}(r)$, the conclusions of Theorem 1.2 simplify to the following corollary.

Corollary 1.1. *Let $c \in (0, \infty)$ and F_c be defined on $(0, 1)$ by*

$$F_c(x) = \mathcal{K}(r) - \ln\left(1 + \frac{c}{\sqrt{1 - x}}\right).$$

Then, the following conclusions hold valid:

(1) *$F_c'(x)$ is absolutely monotonic on $(0, 1)$ if and only if $c \in \left(0, \frac{\pi}{4-\pi}\right]$, where $\frac{\pi}{4-\pi} \approx 2.078$;*
(2) *$F_c''(x)$ is absolutely monotonic on $(0, 1)$ if and only if $c \in (0, \frac{4(2+3\sqrt{4-\pi})}{32-9\pi} - 1]$, where $c_1^* \approx 3.295$;*
(3) *$F_c'''(x)$ is absolutely monotonic on $(0, 1)$ if and only if $c \in (0, c_2^*]$, where $c_2^* \approx 4.64361$ is the unique positive root of the cubic equation:*

$$(75\pi - 256)c^3 + (225\pi - 672)c^2 + (225\pi - 480)c + 75\pi = 0.$$

Remark 1.1. *Corollary 1.1 resolves the open problems in* [5, Problems 5.1 and 5.3] *by providing tight, complete conditions for absolute monotonicity.*

2. Preliminaries and proof of Theorem 1.1

2.1. **Basic knowledge and tools.** In order to prove our result, we need several basic properties of the Gaussian hypergeometric function $F(a,b;c;x)$ for real numbers a,b,c with $-c \notin \mathbb{N}\cup\{0\}$.

(i) Derivative formula

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x). \tag{7}$$

(ii) For $c > a+b$ with $-c \notin \mathbb{N}\cup\{0\}$, we have the value of $x=1$ (cf. [12, p. 49]):

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

In particular,

$$F(a,1-a;2;1) = \frac{\sin(\pi a)}{a(1-a)\pi}. \tag{8}$$

(iii) For $a \in (0,1/2]$, the special asymptotic formula is given by:

$$F(a+1,2-a;3;x) \sim \frac{2\sin(\pi a)}{a(1-a)\pi}\left[R(a) - \frac{1}{a(1-a)} - \ln(1-x)\right] \to \infty \tag{9}$$

as $x \to 1^-$. In particular, by (3),

$$\mathcal{F}_c(1^-) = \frac{R(a)}{2} - \ln c. \tag{10}$$

(iv) Linear transformation formula (cf. [3, Theorem 1.19(10)])

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x). \tag{11}$$

Moreover, the generalized Wallis ratio $\mathcal{W}_{n+1}$ satisfies the recurrence relation

$$\frac{\mathcal{W}_{n+1}}{\mathcal{W}_n} = \frac{\sqrt{(n+a)(n+1-a)}}{n+1}.$$

Specially, it can be written as, for $a = 1/2$,

$$\frac{W_{n+1}}{W_n} = \frac{2n+1}{2(n+1)}. \tag{12}$$

By using the notation $\mathcal{W}_n$, certain specific hypergeometric functions can be simply represented as

$$F(a,1-a;2;x) = \sum_{n=0}^{\infty}\frac{\mathcal{W}_n^2}{n+1}x^n, \tag{13}$$

$$F(a+1,2-a;2;x) = \sum_{n=0}^{\infty}\frac{(n+a)(n+1-a)\mathcal{W}_n^2}{a(1-a)(n+1)}x^n. \tag{14}$$

The key tool is the absolutely monotone rule for the ratio of two power series, which was firstly eastablished in [10].

Proposition 2.1. *Let $p(x) = \sum_{n=0}^{\infty} p_n x^n$ and $q(x) = \sum_{n=0}^{\infty} q_n x^n$ be two power series converging on $(0,r)$ with $p_n > 0$ for all $n \geq 0$. Suppose that the sequence $\{p_{n+1}/p_n\}_{n\geq 0}$ is increasing. Then*

- *if the sequence $\{q_n/p_n\}_{n\geq 0}$ is increasing, then $(q/p)'$ is absolutely monotonic on $(0,r)$;*
- *if the sequence $\{q_n/p_n\}_{n\geq 0}$ is decreasing, then $-(q/p)'$ is absolutely monotonic on $(0,r)$. In particular, $-(1/p)'$ is absolutely monotonic on $(0,r)$.*

From Proposition 2.1, it has been proved that $-(1/p)'$ is absolutely monotonic if $p(x)$ is absolutely monotonic with the increasing of $\{p_{n+1}/p_n\}_{n\geq 0}$. Conversely, if $p(x)>0$ and $-p'(x)$ is absolutely monotonic, we will prove that $1/p(x)$ is absolutely monotonic, which is stated in the following proposition.

Proposition 2.2. *If $p(x)>0$ and $-p'(x)$ is absolutely monotonic on the interval I, then $1/p(x)$ is absolutely monotonic on I.*

Proof. Clearly, $(1/p)'=-p'/p^2>0$. Assume that $(1/p)^{(i)}\geq 0$ for $1\leq i\leq n$. Then, we can prove that

$$\begin{aligned}\left(\frac{1}{p}\right)^{(n+1)} &= \left(-\frac{p'}{p^2}\right)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}(-p')^{(n-k)}\left(\frac{1}{p^2}\right)^{(k)}\\ &= \sum_{k=0}^{n}\binom{n}{k}(-p')^{(n-k)}\left[\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{p}\right)^{(k-j)}\left(\frac{1}{p}\right)^{(j)}\right]\geq 0.\end{aligned}$$

By induction, $(1/p)^{(n)}\geq 0$ for all $n\geq 1$, which in conjunction with $p(x)>0$ implies that $1/p(x)$ is absolutely monotonic on I. ☐

2.2. **The first three steps.** In this subsection, we give the proof of Steps 1, 2 and 3 introduced in Section 1.

Lemma 2.1. (Step 1). *Let B_n be defined in* (5). *Then, $B_0=c/[2(1+c)]$ and B_n satisfies the recurrence relation*

$$(c^2-1)B_n+B_{n-1}=\frac{cW_n}{2(2n-1)}. \tag{15}$$

for $n\geq 1$. In particular, for $n\geq 1$,

$$\begin{aligned} B_{n-1} &= \frac{W_n}{2(2n-1)}, & c&=1,\\ B_{n+1} &= \frac{B_{n-1}}{(c^2-1)^2}+\frac{c[2(c^2-2)n-c^2-1]W_n}{4(c^2-1)^2(n+1)(2n-1)}, & c&\neq 1. \end{aligned} \tag{16}$$

Proof. Let

$$\frac{c}{2(c+\sqrt{1-x})}=\frac{c(c-\sqrt{1-x})}{2(c^2-1+x)}=\sum_{n=0}^{\infty}B_nx^n. \tag{17}$$

This can be rewritten using power series as follows

$$c(c-\sqrt{1-x})=c^2-c\sum_{n=0}^{\infty}\frac{(-1/2)_n}{n!}x^n=2(c^2-1+x)\sum_{n=0}^{\infty}B_nx^n,$$

which is equivalent to

$$c^2-c+\sum_{n=1}^{\infty}\frac{cW_n}{(2n-1)}x^n=c^2-c+2\sum_{n=1}^{\infty}\left[(c^2-1)B_n+B_{n-1}\right]x^n.$$

Hence, equating coefficients of x^n for all $n\geq 1$, we can derive (15). In particular, substituting $c=1$ into (15) yields the first item of (16). For $c\neq 1$, (15) can be rearranged as

$$B_n=-\frac{B_{n-1}}{c^2-1}+\frac{cW_n}{2(c^2-1)(2n-1)},$$

which, together with (12), leads to the following recurrence relation

$$\begin{aligned} B_{n+1} &= -\frac{B_n}{c^2-1} + \frac{cW_{n+1}}{2(c^2-1)(2n+1)} \\ &= \frac{B_{n-1}}{(c^2-1)^2} - \frac{cW_n}{2(c^2-1)^2(2n-1)} + \frac{cW_{n+1}}{2(c^2-1)(2n+1)} \\ &= \frac{B_{n-1}}{(c^2-1)^2} + \frac{c[2(c^2-2)n-c^2-1]W_n}{4(c^2-1)^2(n+1)(2n-1)}. \end{aligned}$$

□

Lemma 2.2. *Let B_n and v_n be defined in* (5)*. Then, $B_n(c)$ is decreasing in c on $[3,\infty)$ and so $v_n(c)$ is increasing in c on $[3,\infty)$ for any $n \geq 1$.*

Proof. Differentiating the right side of (17) with respect to c, we can obtain

$$\frac{1}{2\left[c^2(1-x)^{-1/2}+\sqrt{1-x}+2c\right]} = \frac{1}{2\sum\limits_{n=0}^{\infty}\overline{p}_n x^n} = \sum_{n=0}^{\infty} B_n'(c)x^n, \tag{18}$$

where

$$\overline{p}_n = (c+1)^2, n=0, \left(c^2-\frac{1}{2n-1}\right)W_n, n\geq 1.$$

Clearly, $\overline{p}_n > 0$ for all $n \geq 0$ and $c \in (1,\infty)$. Moreover, we can directly verify that

$$\begin{aligned} &\frac{\overline{p}_2}{\overline{p}_1} - \frac{\overline{p}_1}{\overline{p}_0} = \frac{2+(c-1)(c+5)}{4(c^2-1)} > 0, \\ &\frac{\overline{p}_{n+2}}{\overline{p}_{n+1}} - \frac{\overline{p}_{n+1}}{\overline{p}_n} = \frac{4c^2(n-1)(c^2n+c^2-3)+3(c^2-6)c^2+3}{2(n+2)(n+3)(2c^2n+c^2-1)(2c^2n-c^2-1)} > 0, \quad (n\geq 1) \end{aligned}$$

for $c \in [3,\infty)$, which together with Proposition 2.1 and (18) implies $B_n'(c) < 0$ for $n \geq 1$. The proof of Lemma 2.2 is completed by $v_n(c) = A_n - B_n(c)$. □

Lemma 2.3. (Step 2). *Let $c \in [3,\infty)$ and $v_n = A_n - B_n$ be defined in* (5)*, where $A_n = \frac{a(1-a)}{2\sin(\pi a)} \cdot \frac{\mathcal{W}_n^2}{n+1}$ and B_n is defined by the recurrence in Lemma 2.1. Let $u_n = \sum\limits_{k=0}^{n} v_k$ (partial sum of v_k). Then, there exists a number $a_0 = a_0(c) \in (0,1/2)$ such that exactly one of the following holds for $a \in (0,1/2]$:*

- (*i*) *if $a \in (0,a_0]$, then $v_n \leq 0$ for $n \geq 3$, which implies u_n is decreasing for $n \geq 2$;*
- (*ii*) *if $a \in (a_0,1/2]$, then there exists an integer $n_0 \geq 3$ such that $v_n > 0$ for $3 \leq n \leq n_0$ and $v_n \leq 0$ for $n \geq n_0+1$, which implies u_n is first increasing (for $2 \leq n \leq n_0$) and then decreasing (for $n \geq n_0$) (i.e. u_n is unimodal).*

Proof. The key to characterizing u_n monotonicity is analyzing the sign of v_n (since $u_n - u_{n-1} = v_n$). By (6), we see that $v_n = A_n - B_n$ with $A_n > 0$. If we prove that $\{B_n/A_n\}_{n\geq 3}$ is increasing with $\lim\limits_{n\to\infty}(B_n/A_n) = \infty$ and there exists $a_0 = a_0(c) \in (0,1/2)$ such that

$$\frac{B_3}{A_3} = \frac{B_3(c)}{A_3(a)} \geq 1, \text{if } a \in (0,a_0], \ < 1, \text{if } a \in (a_0,1/2),$$

then the required assertion follows.

(1) Prove that B_n/A_n is increasing for $n \geq 3$.

Define $T_n = B_{n+1} - \frac{A_{n+1}}{A_n} B_n$, we aim to show $T_n > 0$ for $n \geq 3$. By (12) and (15), we rewrite T_n as

$$\begin{aligned} T_n &= -\frac{B_n}{c^2-1} + \frac{cW_{n+1}}{2(c^2-1)(2n+1)} - \frac{(n+a)(n+1-a)}{(n+1)(n+2)} B_n \\ &= -\lambda_n B_n + \frac{cW_n}{4(c^2-1)(n+1)}, \end{aligned} \tag{19}$$

where

$$\lambda_n = \frac{1}{c^2-1} + \frac{(n+a)(n+1-a)}{(n+1)(n+2)}.$$

For $c \in [3, \infty)$, direct substitution of $n = 3$ and $n = 4$ into T_n yields positive expressions

$$\begin{aligned} T_3 &= \frac{c\left[(1-2a+2a^2)(c+1)(c^2+4c+5)+10(c+5)\right]}{1280(c+1)^5} > 0, \\ T_4 &= \frac{c\left[(1-a+a^2)(c+1)(5c^3+25c^2+47c+35)+6(3c^2+18c+35)\right]}{7680(c+1)^6} > 0. \end{aligned}$$

Assume that $T_k > 0$ for all $3 \leq k \leq n$ (where $n \geq 4$). By (19), the assumption $T_{n-1} > 0$ is equivalent to

$$B_{n-1} < \frac{cW_{n-1}}{4\lambda_{n-1}(c^2-1)n}. \tag{20}$$

Then, we can show by (16), (19) and (20) that

$$\begin{aligned} T_{n+1} &= -\lambda_{n+1} B_{n+1} + \frac{cW_{n+1}}{4(c^2-1)(n+2)} \\ &= -\lambda_{n+1}\left[\frac{B_{n-1}}{(c^2-1)^2} + \frac{c\left(2(c^2-2)n-c^2-1\right)W_n}{4(c^2-1)^2(n+1)(2n-1)}\right] + \frac{cW_{n+1}}{4(c^2-1)(n+2)} \\ &> -\lambda_{n+1}\left[\frac{cW_{n-1}}{4\lambda_{n-1}(c^2-1)^3 n} + \frac{c\left(2(c^2-2)n-c^2-1\right)W_n}{4(c^2-1)^2(n+1)(2n-1)}\right] + \frac{cW_{n+1}}{4(c^2-1)(n+2)} \\ &= \frac{c\left[\tau_0+\tau_1(n-2)+\tau_2(n-2)^2+\tau_3(n-2)^3+2c^2(c^2-2)(n-2)^4\right]W_n}{8(c^2-1)^2(n+1)(n+2)(n+3)(2n-1)\left[\begin{array}{l} 4-a+a^2+(2-a)(1+a)c^2 \\ +(2+3c^2)(n-2)+c^2(n-2)^2 \end{array}\right]} \\ &> 0 \end{aligned}$$

for $n \geq 3$, where

$$\begin{aligned} \tau_0 &= \left[\begin{array}{l} 3(1-2a)(4+73a+75a^2+3a^3)+477a^3+90(c^2-4) \\ +3(2-a)(1+a)\left(2(2a-1)^2(c^2-4)+(1-2a+2a^2)(c^2-4)^2\right) \end{array}\right], \\ \tau_1 &= \frac{1}{4}\left[\begin{array}{l} 765(c^2-3)+54(c^2-4)^2+(2a-1)^2(43+24a-24a^2) \\ +(2a-1)^2\big((143+20a-20a^2)(c^2-4)+2(c^2-4)^2(11+2a-2a^2)\big), \end{array}\right], \\ \tau_2 &= 191+(c^2-4)(89+18c^2)+(2a-1)^2\left[27+(c^2-4)(9+4c^2)\right], \\ \tau_3 &= 84+(c^2-4)(27+10c^2)+(2a-1)^2c^2(c^2-2). \end{aligned}$$

By induction, we obtain $T_n > 0$ for all $n \geq 3$, hence B_n/A_n is increasing for $n \geq 3$.

(2) Compute $\lim_{n\to\infty}(B_n/A_n) = \infty$.

By (15) and $W_n \sim 1/\sqrt{n\pi}$ as $n \to \infty$, it is readily got that

$$B_n \sim \frac{1}{4c\sqrt{\pi}n^{3/2}}.$$

Also,

$$\begin{aligned} A_n &= \frac{a(1-a)\pi}{2\sin(\pi a)}\frac{(a)_n(1-a)_n}{n!(n+1)!} = \frac{a(1-a)}{2}\frac{\Gamma(n+a)\Gamma(n+1-a)}{\Gamma(n+1)\Gamma(n+2)} \\ &\sim \frac{a(1-a)}{2}n^{a-1}n^{1-a-2} = \frac{a(1-a)}{2}\frac{1}{n^2}. \end{aligned}$$

This deduces

$$\frac{B_n}{A_n} \sim \frac{1}{2c\sqrt{\pi}a(1-a)}\sqrt{n} \to \infty \quad \text{as} \quad n \to \infty.$$

(3) Determine the threshold a_0.

We analyze $B_3(c)/A_3(a)$ as a function of $a \in (0, 1/2]$ for $c \in [3, \infty)$. The function

$$A_3(a) = \frac{a^2(3-a)(1-a)(1-a^2)(4-a^2)\pi}{288\sin(\pi a)}$$

is positive and increasing for $a \in (0, 1/2]$ by L'Hopital Monotone Rule and so $B_3(c)/A_3(a)$ is decreasing for $a \in (0, 1/2]$. Moreover, as $a \to 0^+$, $A_3(a) \to 0$, so $B_3(c)/A_3(0^+) = \infty$. At $a = 1/2$, direct computation gives

$$\begin{aligned} \frac{B_3(c)}{A_3(1/2)} &= \frac{256c(5+4c+c^2)}{25\pi(c+1)^4} < \frac{256c(5+4c+c^2)}{78(c+1)^4} \\ &= 1 - \frac{(c-3)\left[2816+(c-3)(943+262c+39c^2)\right]}{39(c+1)^4} < 1. \end{aligned}$$

By the Intermediate Value Theorem, there exists a unique $a_0 = a_0(c) \in (0, 1/2)$ such that $B_3(c)/A_3(a) \geq 1$ for $a \in (0, a_0]$ and $B_3(c)/A_3(a) < 1$ for $a \in (a_0, 1/2]$.

This completes the proof of Lemma 2.4 via (1)-(3). □

Lemma 2.4. (Step 3-1). *For any fixed $n \geq 0$, $u_n(c)$ is decreasing in c on $(0, \infty)$ with $u_n(0) > 0$ and $u_n(\infty) < 0$, and we denote by c_n the unique positive zero of $u_n(c)$.*

Proof. Differentiation of (6) with respect to c yields

$$-\frac{1}{2\sqrt{1-x}}\left(\frac{1}{c+\sqrt{1-x}}\right)^2 = \sum_{n=0}^{\infty} u_n'(c)x^n. \tag{21}$$

Clearly,

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} W_n x^n \quad \text{and} \quad \frac{1}{c+\sqrt{1-x}} = \frac{1}{c+1-\sum_{n=1}^{\infty}\frac{W_n}{2n-1}x^n}$$

are both absolutely monotonic on $(0, 1)$, by Proposition 2.2. This together with (21) deduces $u_n'(c) \leq 0$ for $n \geq 0$.

Moreover, taking $c \to 0$ and $c \to \infty$ in (6), it follows from (14) that

$$\begin{aligned} \mathcal{F}_0'(x) &= \frac{a(1-a)\pi}{2\sin(\pi a)}F(a+1, 2-a; 2; x) \\ &= \sum_{n=0}^{\infty}\frac{(n+a)(n+1-a)\pi\mathcal{W}_n^2}{2\sin(\pi a)(n+1)} = \sum_{n=0}^{\infty} u_n(0)x^n, \\ \mathcal{F}_\infty'(x) &= \frac{a(1-a)\pi}{2\sin(\pi a)}F(a+1, 2-a; 2; x) - \frac{1}{2(1-x)} \\ &= \sum_{n=0}^{\infty}\left[\frac{(n+a)(n+1-a)\pi\mathcal{W}_n^2}{2\sin(\pi a)(n+1)} - \frac{1}{2}\right]x^n = \sum_{n=0}^{\infty} u_n(\infty)x^n. \end{aligned}$$

Clearly, $u_n(0) > 0$ and

$$u_n(\infty) = \frac{(n+a)(n+1-a)\pi\mathcal{W}_n^2}{2\sin(\pi a)(n+1)} - \frac{1}{2} < \lim_{n\to\infty} \frac{(n+a)(n+1-a)\pi\mathcal{W}_n^2}{2\sin(\pi a)(n+1)} - \frac{1}{2} = 0,$$

since $\left[(n+a)(n+1-a)\pi\mathcal{W}_n^2\right] / [2\sin(\pi a)(n+1)]$ is strictly increasing for $n \geq 0$. This completes the proof. □

Lemma 2.5. (Step 3-2). *Let c_n denote the unique positive zero of $u_n(c)$ for each $n \geq 0$. Then, $c_n > 3$ for all $n \geq 0$.*

Proof. The conclusion of Lemma 2.5 is valid if we can first prove that $u_k(3) > 0$ for $k = 0, 1, 2$ and all $a \in (0, 1/2]$.

Suppose this assertion (i.e. $u_k(3) > 0$ for $k = 0, 1, 2$) holds. By Lemma 2.3, for any $n \geq 0$, the sequence $u_n(3)$ satisfies $u_n(3) > \min\{u_0(3), u_1(3), u_2(3), u_\infty(3)\} = 0$ for all $n \geq 0$. Here, $u_\infty(3) = 0$ can be derived from three facts:

- $u_\infty(3) = \sum_{k=0}^{\infty} v_k$ (by the definition of u_n as the partial sum of v_k);
- $\sum_{k=0}^{\infty} v_k = \mathcal{G}_3(1^-)$ (from the power series representation of $\mathcal{G}_c(x)$ in (5));
- $\mathcal{G}_3(1^-) = 0$ (by (5) and the asymptotic properties of $\mathcal{G}_c(x)$ at $x \to 1^-$ in (8)).

Thus, $u_n(3) > 0$ for all $n \geq 0$. Combining this with Lemma 2.4, we infer that the unique positive root c_n of $u_n(c)$ must satisfy $c_n > 3$ (since $u_n(3) > 0$ and $u_n(c)$ decreases to $-\infty$ as $c \to \infty$).

Next, we verify $u_k(3) > 0$ for $k = 0, 1, 2$ and all $a \in (0, 1/2]$:

Recall the inequality established in [21, Lemma 2.6] for $a \in (0, 1/2]$:

$$\frac{\pi a}{\sin(\pi a)} > \frac{a^2 - a + 1}{1 - a} \tag{22}$$

holds for $a \in (0, 1/2]$. By the definition of $u_n(c)$ (from (6) and Step 1) and (22), it can be verified directly that

$$\begin{aligned}
u_0(3) &= \frac{(1-a)\pi a}{2\sin(\pi a)} - \frac{3}{8} > \frac{(1-a)}{2}\frac{a^2-a+1}{1-a} - \frac{3}{8} = \frac{(1-2a)^2}{8} \geq 0, \\
u_1(3) &= \frac{(2-a)(1-a^2)\pi a}{4\sin(\pi a)} - \frac{27}{64} > \frac{(2-a)(1-a^2)}{4}\frac{a^2-a+1}{1-a} - \frac{27}{64} \\
&= \frac{(1-2a)^2(5+4a-4a^2)}{64} \geq 0, \\
u_2(3) &= \frac{(3-a)(4-a^2)(1-a^2)\pi a}{24\sin(\pi a)} - \frac{225}{512} > \frac{(3-a)(4-a^2)(1-a^2)}{24}\frac{a^2-a+1}{1-a} - \frac{225}{512} \\
&= \frac{(1-2a)^2\left[(1-2a)(93+302a+504a^2+976a^3)+1968a^4\right]}{1536} \geq 0
\end{aligned}$$

for $a \in (0, 1/2]$. In conclusion, $u_k(3) > 0$ for $k = 0, 1, 2$ and all $a \in (0, 1/2]$, and so $c_n > 3$ for all $n \geq 0$. □

Lemma 2.6. (Step 3-3). *Let c_n denote the unique positive zero of $u_n(c)$ for each $n \geq 0$. Then, $c_0 < c_1 < c_2$ holds.*

Proof. By Lemma 2.4, $u_n(c)$ is strictly decreasing in c on $(0, \infty)$ for each fixed $n \geq 0$. If we prove that

$$u_1(c_0) > 0 = u_1(c_1) \quad \text{and} \quad u_2(c_1) > 0 = u_2(c_2),$$

then $c_0 < c_1 < c_2$ holds.

(1): Prove $u_1(c_0) > 0$.

First, we derive the explicit expression of c_0 using the definition of $u_0(c)$. From (6) and the power series construction of u_n in Step 1, $u_0(c)$ and $u_1(c)$ are defined as

$$u_0(c) = \frac{a(1-a)\pi}{2\sin(\pi a)} - \frac{c}{2(c+1)}, \quad u_1(c) = \frac{a(1-a^2)(2-a)\pi}{4\sin(\pi)} - \frac{c(2c+3)}{4(c+1)^2}.$$

Since c_0 is the root of $u_0(c) = 0$, we solve it for c as

$$c_0 = \frac{a(1-a)\pi}{\sin(\pi a) - a(1-a)\pi}.$$

Next, substituting $c = c_0$ into $u_1(c)$, and using the inequality (22) infer that

$$u_1(c_0) = \frac{a(1-a)^2\pi}{4\sin(\pi a)}\left[\frac{\pi a}{\sin(\pi a)} - \frac{a^2-a+1}{1-a}\right] > 0.$$

(2): Prove $u_2(c_1) > 0$.

First, we use (22) to derive a lower bound for c_1. Substitute $c = \frac{1-a+a^2}{a(1-a)}$ into $u_1(c)$, and simplify using (22):

$$u_1\left(\frac{1-a+a^2}{a(1-a)}\right) = \frac{(1-a^2)(2-a)}{4}\left[\frac{\pi a}{\sin(\pi a)} - \frac{a^2-a+1}{1-a}\right] > 0.$$

Since $u_1(c)$ is strictly decreasing (Lemma 2.4) and $u_1(c_1) = 0$, this implies

$$c_1 > \frac{1-a+a^2}{a(1-a)} = 3 + \frac{(2a-1)^2}{a(1-a)} \geq 3.$$

From Lemma 2.2, $v_n(c)$ is increasing in c on $[3, \infty)$. By this and $c_1 > 3$, simplifying by (22) yields

$$\begin{aligned} v_2(c_1) &\geq v_2\left(\frac{1-a+a^2}{a(1-a)}\right) \\ &= \frac{a(1+a)(1-a)^2(2-a)}{24}\left[\frac{\pi a}{\sin(\pi a)} - \frac{3(1-a+a^2)(1+2a-2a^2)}{2(1-a^2)(2-a)}\right] \\ &> \frac{a(1+a)(1-a)^2(2-a)}{24}\left[\frac{a^2-a+1}{1-a} - \frac{3(1-a+a^2)(1+2a-2a^2)}{2(1-a^2)(2-a)}\right] \\ &= \frac{1}{48}a(1-a)(1-2a)^2(1-a+a^2) \geq 0 \end{aligned}$$

and $u_2(c_1) = u_2(c_1) - u_1(c_1) = v_2(c_1) > 0$ (from the relationship between u_n and v_n in Step 1 and the definition of c_1).

This completes the proof. □

2.3. **Step 4: Proof of Theorem 1.1.** In this subsection, by means of Lemmas 2.4, 2.5, 2.6 (Step 3), we can complete the proof of our main result.

Proof. From (6), $\mathcal{F}_c'$ can be decomposed using the auxiliary function $\mathcal{G}_c(x)$, leading to the power series expansion:

$$\mathcal{F}_c'(x) = \frac{\mathcal{G}_c(x)}{1-x} = \sum_{n=0}^{\infty} u_n(c)x^n, \tag{23}$$

where $u_n(c) = \sum_{k=0}^{n} v_k(c)$ and $v_k(c) = A_k - B_k(c)$. Here, $A_k = \frac{a(1-a)\pi}{2\sin(\pi a)}\frac{\mathcal{W}_k^2}{k+1}$ with the generalized Wallis ratio $\mathcal{W}_k$ and B_k is defined by the recurrence in Lemma 2.1.

Necessity. Assume $\mathcal{F}_c^{(k+1)}$ is absolutely monotonic on $(0,1)$. By the absolute monotonicity criterion, all coefficients of its Maclaurin series are non-negative specifically, the leading coefficient $u_k(c) \geq 0$ by (23).

From Lemma 2.4 (Step 3-1), $u_n(c)$ is strictly decreasing in c on $(0,\infty)$, with $u_n(0) > 0$ and $u_n(\infty) < 0$. Thus, $u_k(c) \geq 0$ if and only if $c \leq c_k$ (where c_k is the unique positive root of $u_k(c) = 0$). Since $c > 0$ (as $\ln\left(1 + \frac{c}{\sqrt{1-x}}\right)$ is well-defined for positive c), we conclude $c \in (0, c_k]$.

Sufficiency. As shown in Lemma 2.5, $c_k > 3$ for all $k \geq 0$. We will divide into two cases $[3, c_k]$ and $(0,3)$ to complete the proof.

Case 1: $c \in [3, c_k]$.

By Lemma 2.4, $u_n(c)$ is strictly decreasing in c, so $u_k(c) \geq u_k(c_k) = 0$ (since $c \leq c_k$). We now verify $u_n(c) \geq 0$ for all $n \geq k$ using Lemma 2.3 (monotonicity of u_n):

- $k = 0$. By Lemma 2.6, $c_0 < c_1 < c_2$, so $u_0(c) \geq 0$, $u_1(c) \geq 0$, $u_2(c) \geq 0$ for $c \in [3, c_0]$. Hence, $u_n(c) \geq \min\{u_2(c), u_\infty\} = 0$ for $n \geq 3$ by Lemma 2.3 ($u_\infty(c) = \lim_{n\to\infty} u_n(c) = 0$ from the asymptotic property of $\mathcal{G}_c(x)$ at $x \to 1^-$).
- $k = 1$. By Lemma 2.6, $c_1 < c_2$, so $u_1(c) \geq 0$ and $u_2(c) \geq 0$ for $c \in [3, c_0]$. By Lemma 2.3, $u_n(c) > 0$ for $n \geq 3$ (since $u_2(c) > 0$ and $u_n(c)$ is unimodal or decreasing).
- $k \geq 2$. By Lemma 2.3, $u_n(c)$ is either unimodal (first increasing, then decreasing) or decreasing for$n \geq 2$. Thus, $u_n(c) \geq \min\{u_k, u_\infty\} = 0$ for all $n \geq k$ (since $u_k(c) \geq 0$ and $u_\infty(c) = 0$).

In all subcases, $u_n(c) \geq 0$ for all $n \geq k$, so $\mathcal{F}_c^{(k+1)}$ $(k \geq 0)$ is absolutely monotonic on $(0,1)$.

Case 2: $c \in (0,3)$.

We rewrite $\mathcal{F}_c(x)$ using $\mathcal{F}_3(x)$ (the function at $c = 3$) to leverage the known absolute monotonicity of $\mathcal{F}_3^{(k+1)}(x)$:

$$\mathcal{F}_c(x) = \mathcal{F}_3(x) + \log\left(1 + \frac{3}{\sqrt{1-x}}\right) - \log\left(1 + \frac{c}{\sqrt{1-x}}\right).$$

Differentiate both sides to relate $\mathcal{F}_c'(x)$ and $\mathcal{F}_3'(x)$:

$$\mathcal{F}_c'(x) = \mathcal{F}_3'(x) + \frac{3-c}{2\sqrt{1-x}(3+\sqrt{1-x})(c+\sqrt{1-x})}. \tag{24}$$

By Case 1, $\mathcal{F}_3'$ is absolutely monotonic on $(0,1)$ and so is $\mathcal{F}_3^{(k+1)}$ $(k \geq 0)$. For $c \in (0,3)$, $3-c > 0$, by Proposition 2.2, the functions $1/\sqrt{1-x}$, $1/(3+\sqrt{1-x})$ and $1/(c+\sqrt{1-x})$ are all absolutely monotonic on $(0,1)$–their product (and scalar multiple by $\frac{3-c}{2}$)

$$\frac{3-c}{2} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{3+\sqrt{1-x}} \cdot \frac{1}{c+\sqrt{1-x}}$$

is also absolutely monotonic. Differentiating this term k times preserves absolute monotonicity.

Since the sum of two absolutely monotonic functions is absolutely monotonic, by (24), $\mathcal{F}_c^{(k+1)}(x)$ is absolutely monotonic on $(0,1)$.

This completes the proof. □

Remark 2.1. *The method used to prove the absolute monotonicity of* $\mathcal{F}_c^{(k+1)}(x)$ *for* $c \in (0,3)$ *(Case 2 in the proof of Theorem 1.1) can be directly applied to refine the conclusions in [5, Theorem 1.1]. Specifically, [5, Theorem 1.1] only established sufficient conditions for the absolute monotonicity of* $F_c'(x)$ *and* $F_c''(x)$ *with* $c \geq \sqrt{3} \approx 1.732$. *By extending the valid range of c from* $[3, c_k]$ *(as shown in Theorem 1.1), we can derive the necessary and sufficient conditions for* F_c' *and* F_c''.

Remark 2.2. *Lemma 2.6 has established the strict monotonicity of c_n for the first three non-negative integers, i.e. $c_0 < c_1 < c_2$, (where c_n denotes the unique positive root of $u_n(c) = 0$). A notable generalization of this result is that the strict monotonicity of c_n holds for all non-negative integers n specifically, $c_{n-1} < c_n$ for every integers $n \geq 1$. The proof of this generalization relies on the monotonicity of $u_n(c)$ (Lemma 2.4) and Theorem 1.1. For any $n \geq 1$, by Theorem 1.1, $F^{(n)}_{c_{n-1}}$ is absolutely monotonic on $(0,1)$, all coefficients of its Maclaurin series must be non-negative–including the key coefficient $u_n(c_{n-1}) > 0$. Since $u_n(c_{n-1}) > 0 = u_n(c_n)$, the strict monotonicity of $u_n(c)$ implies $c_{n-1} < c_n$ for $n \geq 1$.*

3. New functional inequalities related to $\mathcal{K}_a(r)$

In this section, Theorem 1.1 will be applied to establish several functional inequalities involving the generalized elliptic integral of the first kind $\mathcal{K}_a(r)$.

By Theorem 1.2, we can extend the property of absolute monotonicity to $\mathcal{F}_c$.

Theorem 3.1. *Let $c \in (0, \infty)$ and $\mathcal{F}_c(x)$ be defined in (4). Then, $\mathcal{F}_c(x)$ is absolutely monotonic on $(0,1)$ if and only if $c \in (0, \frac{a(1-a)\pi}{\sin(\pi a) - a(1-a)\pi}]$.*

Before proving Theorem 3.1, we need the following lemma.

Lemma 3.1. *For $a \in (0, 1/2]$, the following inequality holds:*

$$\exp\left(\frac{\pi}{2\sin(\pi a)}\right) > \frac{\sin(\pi a)}{\sin(\pi a) - a(1-a)\pi}, \tag{25}$$

where $\exp(\cdot)$ *denotes the exponential function.*

Proof. To prove Lemma 3.1, we divide into two cases to complete the proof.

Case 1: $a \in (0, 0.3]$.

In this case, we first show that the inequality

$$\frac{\pi}{\sin(\pi a)} < \frac{1 - 2a^2}{a(1-a)} \tag{26}$$

holds. By the power series expansion in [1, Ch.4.3., p.68] we write

$$\begin{aligned}
\eta(a) &= \frac{\pi}{\sin(\pi a)} - \frac{1-2a^2}{a(1-a)} \\
&= \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)|B_{2n}|\pi^{2n}}{(2n)!} a^{2n-1} - \left(\frac{1}{a} + 1 - \sum_{n=1}^{\infty} a^n\right) \\
&= -1 + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)|B_{2n}|\pi^{2n}}{(2n)!} a^{2n-1} + \sum_{n=1}^{\infty} a^n,
\end{aligned}$$

which implies $\eta'(a) > 0$ and thereby $\eta(a) \leq \eta(0.3) = -0.0215398\cdots < 0$. This completes the proof of (26).

Let

$$g(a) = \left[1 - \frac{a(1-a)\pi}{\sin(\pi a)}\right] \exp\left(\frac{\pi}{2\sin(\pi a)}\right) - 1.$$

Then differentiation yields

$$g'(a) = -\frac{\pi}{2\sin(\pi a)} \exp\left(\frac{\pi}{2\sin(\pi a)}\right) g_0(a) \tag{27}$$

and by (26)

$$\begin{aligned} g_0(a) &= 2-4a-\frac{\pi}{\tan(\pi a)}\left[2a(1-a)-1+\frac{a(1-a)\pi}{\sin(\pi a)}\right] \\ &> 2-4a-\frac{\pi}{\tan(\pi a)}\left[2a(1-a)-1+1-2a^2\right] \\ &= 2(1-2a)\left[1-\frac{\pi a}{\tan(\pi a)}\right] \geq 0 \end{aligned}$$

for $a \in (0,1/2]$. This in combination with (27) implies that $g(a)$ is decreasing on $(0,0.3]$ and so $g(a) \geq g(0.3) = 0.286122\cdots > 0$, equivalently, (25) holds for $a \in (0,0.3]$.

Case 2: $a \in (0.3, 0.5]$.

Let

$$h_1(a) = \exp\left(\frac{\pi}{2\sin(\pi a)}\right), \quad h_2(a) = \frac{\sin(\pi a)}{\sin(\pi a)-a(1-a)\pi}.$$

Then, it is easy to show that $h_1(a)$ and $h_2(a)$ are decreasing on $(0,1/2]$ by L'Hopital Monotone Rule. Numerically, we directly verify $h_1(a)-h_2(a)>0$ on the following four intervals:

$$\begin{aligned} h_1(a)-h_2(a) &\geq h_1(0.35)-h_2(0.3) = 0.410218\cdots > 0, \quad a \in [0.3,0.35], \\ h_1(a)-h_2(a) &\geq h_1(0.4)-h_2(0.35) = 0.161401\cdots > 0, \quad a \in [0.35,0.4], \\ h_1(a)-h_2(a) &\geq h_1(0.45)-h_2(0.4) = 0.079717\cdots > 0, \quad a \in [0.4,0.45], \\ h_1(a)-h_2(a) &\geq h_1(0.5)-h_2(0.45) = 0.110426\cdots > 0, \quad a \in [0.45,0.5]. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Now we are in a position to prove Theorem 3.1.

Proof. By (6), we can write

$$\mathcal{F}_c(x) = \mathcal{F}_c(0) + \sum_{n=0}^{\infty}\frac{u_n}{n+1}x^{n+1} = \frac{\pi}{2\sin(\pi a)} - \ln(1+c) + \sum_{n=1}^{\infty}\frac{u_{n-1}}{n}x^n. \tag{28}$$

The necessary condition requires that $\mathcal{F}_c(0) \geq 0$ and $u_0(c) \geq 0$, which gives

$$0 < c \leq \exp\left(\frac{\pi}{2\sin(\pi a)}\right)-1 \quad \text{and} \quad 0 < c \leq c_0 = \frac{\sin(\pi a)}{\sin(\pi a)-a(1-a)\pi}-1$$

and so by (25)

$$0 < c \leq \frac{a(1-a)\pi}{\sin(\pi a)-a(1-a)\pi}.$$

Conversely, if $c \in (0, \frac{a(1-a)\pi}{\sin(\pi a)-a(1-a)\pi}]$, then by (25) and Theorem 1.2, we see that $\mathcal{F}_c(0) > 0$ and $u_n \geq 0$ for $n \geq 0$. This together with (28) gives the absolute monotonicity of $\mathcal{F}_c$. □

Remark 3.1. *Although $\mathcal{F}_c$ is not absolutely monotonic on $(0,1)$ for $c \in (c_0,\infty)$, the monotonicity of u_n allows us to demonstrate that there exists sufficiently large $N > 0$ such that $u_n \geq u_\infty = 0$ for $n \geq N$ if $c \in (c_0,\infty)$. This in conjunction with Theorem 3.1 gives a negative answer to* [5, Problem 5.3]*, that is, $-\mathcal{F}_c$ never will be absolutely monotonic on $(0,1)$ for any $c \in (0,\infty)$.*

Proposition 3.1. *Let u_n be defined in* (6) *and*

$$\mathcal{Q}_{n,c}(x) = \frac{1}{x^{n+1}}\left[\mathcal{K}_a(\sqrt{x}) - \ln\left(1+\frac{c}{\sqrt{1-x}}\right) - \frac{\pi}{2\sin(\pi a)} + \ln(1+c) - s_n(x)\right]$$

for $n \geq 0$, *where* $s_0(x) = 0$ *and* $s_n(x) = \sum_{k=1}^{n}(u_{k-1}/k)x^k$ *for* $n \geq 1$. *Then,* $\mathcal{Q}_{n,c}(x)$ *is absolutely monotonic on* $(0,1)$ *if and only if* $c \in (0, c_n]$, *where* c_n *is the unique positive root of* $u_n(c) = 0$. *Consequently, if* $c \in (0, c_n]$, *then the double inequality*

$$pr^{2n+2} < \mathcal{K}_a(r) - \left[\ln\left(1 + \frac{c}{r'}\right) + \frac{\pi}{2\sin(\pi a)} - \ln(1+c) + s_n(r^2)\right] < qr^{2n+2} \tag{29}$$

holds for all $r \in (0,1)$ *if and only if* $p \leq u_n/(n+1)$ *and*

$$q \geq \frac{R(a)}{2} - \frac{\pi}{2\sin(\pi a)} + \ln\left(1 + \frac{1}{c}\right) - s_n(1).$$

Proof. As shown in (28), for any $n \geq 0$, $\mathcal{Q}_{n,c}$ can be written as

$$\mathcal{Q}_{n,c}(x) = \frac{1}{x^{n+1}}\Big[\mathcal{F}_c(x) - \mathcal{F}_c(0) - s_n(x)\Big] = \sum_{k=0}^{\infty} \frac{u_{k+n}}{k+n+1} x^k. \tag{30}$$

Then $\mathcal{Q}_{n,c}(x)$ is absolutely monotonic on $(0,1)$ if and only if $u_{k+n} \geq 0$ for all $k \geq 0$, equivalently, $c \in (0, c_n]$ by Lemma 2.4. Consequently, if $c \in (0, c_n]$, then $\mathcal{Q}_{n,c}$ is absolutely monotonic and so is increasing on $(0,1)$. By (3), the double inequality

$$\mathcal{Q}_{n,c}(0^+) < \mathcal{Q}_{n,c}(c) < \mathcal{Q}_{n,c}(1^-)$$

holds for $x \in (0,1)$, where

$$\mathcal{Q}_{n,c}(0^+) = \frac{u_n}{n+1}, \quad \mathcal{Q}_{n,c}(1^-) = \frac{R(a)}{2} - \frac{\pi}{2\sin(\pi a)} + \ln\left(1 + \frac{1}{c}\right) - s_n(1).$$

This gives (29) by substituting $x = r^2$. □

Let

$$\mathcal{L}_n(c,r) = \ln\left(1 + \frac{c}{r'}\right) + \frac{\pi}{2\sin(\pi a)} - \ln(1+c) + s_{n+1}(r^2), \tag{31}$$

$$\begin{aligned}\mathcal{U}_n(c,r) = {} & \ln\left(1 + \frac{c}{r'}\right) + \frac{\pi}{2\sin(\pi a)} - \ln(1+c) + s_n(r^2) \\ & + \left[\frac{R(a)}{2} - \frac{\pi}{2\sin(\pi a)} + \ln\left(1 + \frac{1}{c}\right) - s_n(1)\right] r^{2n+2}.\end{aligned} \tag{32}$$

Then differentiation with respect to c by Lemma 2.4 yields

$$\begin{aligned}\mathcal{L}_{n+1}'(c,r) - \mathcal{L}_n'(c,r) &= \Big[\mathcal{L}_{n+1}(c,r) - \mathcal{L}_n(c,r)\Big]_c' = \frac{u_{n+1}'(c)}{n+2} r^{2n+4} < 0, \\ \mathcal{U}_{n+1}'(c,r) - \mathcal{U}_n'(c,r) &= \left[\frac{\pi}{2\sin(\pi a)} - \frac{R(a)}{2} - \ln\left(1 + \frac{1}{c}\right) + \sum_{k=1}^{n+1} \frac{u_{k-1}(c)}{k}\right]_c' r'^2 r^{2n+2} \\ &= \left[\sum_{k=1}^{n+1} \frac{u_{k-1}'(c)}{k} + \frac{1}{c(1+c)}\right] r'^2 r^{2n+2} \\ &\leq \left[u_0'(c) + \frac{1}{c(1+c)}\right] r'^2 r^{2n+2} = -\frac{(3c+2)r'^2 r^{2n+2}}{2c(c+1)^2} < 0,\end{aligned}$$

which in combination with (28) implies

$$\mathcal{L}_n'(c,r) > \mathcal{L}_{n+1}'(c,r) > \cdots > \mathcal{L}_\infty'(c,r) = \left[\ln\left(1 + \frac{c}{r'}\right) + \mathcal{F}_c(r^2)\right]_c' = \Big[\mathcal{K}_a(r)\Big]_c' = 0$$

and

$$\begin{aligned}\mathcal{U}_n'(c,r)<\mathcal{U}_{n-1}'(c,r)<\cdots<\mathcal{U}_0'(c,r)&=\left[\ln\left(1+\frac{c}{r'}\right)-\ln(1+c)+\ln\left(1+\frac{1}{c}\right)r^2\right]_c'\\&=-\frac{r'[c(1-r')+r^2]}{c(c+1)(c+r')}<0\end{aligned}$$

for $n\geq 0$. That is to say, $\mathcal{L}_n(c,r)$ is increasing and $\mathcal{U}_n(c,r)$ is decreasing for $c\in(0,\infty)$ with fixed $n\geq 0$.

We now establish the following chain of inequalities.

Proposition 3.2. *Let $c\in(0,c_n]$ and $\mathcal{L}_n(c,r)$, $\mathcal{U}_n(c,r)$ be given in (31) and (32), where c_n is the unique positive root of $u_n(c)=0$. Then, the chain of inequalities*

$$\begin{aligned}\mathcal{L}_0(c_0,r)<\cdots<\mathcal{L}_{n-1}(c_{n-1},r)<\mathcal{L}_n(c_n,r)<\mathcal{K}_a(r)\\<\mathcal{U}_n(c_n,r)<\mathcal{U}_{n-1}(c_{n-1},r)<\cdots<\mathcal{U}_0(c_0,r)\end{aligned}\tag{33}$$

holds for all $r\in(0,1)$ with $n\geq 0$.

Proof. By Proposition 3.1, it follows from (31) and (32) that

$$\mathcal{L}_n(c_n,r)<\mathcal{K}_a(r)<\mathcal{U}_n(c_n,r)\tag{34}$$

holds for all $r\in(0,1)$. From Remark 2.2, it has been shown that the roots c_n satisfy strict monotonicity: $c_{n-1}\leq c_n$ for all $n\geq 1$. By using the monotonicity of $\mathcal{L}_n(c,r)$ and $\mathcal{U}_n(c,r)$ with respect to c, it is readily arrived at

$$\mathcal{L}_n(c_{n-1},r)<\mathcal{L}_n(c_n,r),\quad \mathcal{U}_n(c_n,r)<\mathcal{U}_n(c_{n-1},r).\tag{35}$$

Moreover, for $n\geq 1$, by (10),

$$\begin{aligned}\mathcal{L}_n(c_{n-1},r)-\mathcal{L}_{n-1}(c_{n-1},r)&=\frac{u_n(c_{n-1})}{n+1}r^{2n+2}>0,\\\mathcal{U}_n(c_{n-1},r)-\mathcal{U}_{n-1}(c_{n-1},r)&=\left[\mathcal{F}_{c_{n-1}}(0)+\sum_{k=1}^{n}\frac{u_{k-1}(c_{n-1})}{k}-\frac{R(a)}{2}+\ln c_{n-1}\right]r'^2r^{2n}\\&<\left[\mathcal{F}_{c_{n-1}}(1^-)-\frac{R(a)}{2}+\ln c_{n-1}\right]r'^2r^{2n}=0,\end{aligned}$$

where the inequalities follow from $u_{n+m}(c_{n-1})>0$ for all $m\geq 0$. Combining these results with (34) and (35), we obtain the chain of inequalities (33). ☐

Taking $n=2$ into Proposition 3.2, we obtain the following corollary.

Corollary 3.1. *Let $c\in(0,c_2]$ and $\mathcal{L}_n(c,r)$, $\mathcal{U}_n(c,r)$ be given in (31) and (32). Then, the inequalities*

$$\mathcal{L}_0(c_0,r)<\mathcal{L}_1(c_1,r)<\mathcal{L}_2(c_2,r)<\mathcal{K}_a(r)<\mathcal{U}_2(c_2,r)<\mathcal{U}_1(c_1,r)<\mathcal{U}_0(c_0,r)$$

holds for all $r\in(0,1)$. In particular, for $a=1/2$, the following holds for all $r\in(0,1)$:

$$L_0(c_0^*,r)<L_1(c_1^*,r)<L_2(c_2^*,r)<\mathcal{K}(r)<U_2(c_2^*,r)<U_1(c_1^*,r)<U_0(c_0^*,r),\tag{36}$$

where $L_n(c,r)$, $U_n(c,r)$ and c_n^* coincide with $\mathcal{L}_n(c,r)$, $\mathcal{U}_n(c,r)$ and c_n when $a=1/2$. More precisely,

$$c_0^*=\frac{\pi}{4-\pi},\quad c_1^*=\frac{4(2+3\sqrt{4-\pi})}{32-9\pi}-1,\quad c_2^*=4.64361\cdots.$$

Remark 3.2. *Inequalities*

$$L_1(\kappa_1, r) < L_2(\kappa_0, r) < \mathcal{K}(r) < U_2(\kappa_0, r) < U_1(\kappa_1, r)$$

for all $r \in (0,1)$ *were established in* [5, (4.4)]. *Our result* (36) *refines these inequalities, as* $c_2^* > \kappa_0$.

4. Conclusions

This study investigates the absolute monotonicity of arbitrary-order derivatives of a function constructed from the generalized elliptic integral of the first kind and a logarithmic asymptotic function. We establish the necessary and sufficient conditions for the parameter in the logarithmic term, precisely identifying its optimal range to ensure absolute monotonicity on the interval $(0,1)$. This result fully resolves two key problems from previous research, providing an exact solution to one and a negative answer to the conjecture about the absolute monotonicity of the functions negative. The proposed methodology, based on power series analysis and sequence monotonicity, overcomes limitations of traditional methods and is extendable to zero-balanced hypergeometric functions. Additionally, we derive new functional inequalities involving the generalized elliptic integral of the first kind, forming nested chains of tight bounds adjustable for precision. These findings enrich the analytical properties of generalized elliptic integrals and offer new tools for related fields. Future research can explore applications of this method in more general special functions and complex domains.

5. Acknowledgement

The author would like to express sincere gratitude to the anonymous referees for their meticulous reviews and valuable suggestions that significantly improved the quality of this manuscript. This work was supported by the National Natural Science Foundation of China (Grant no. 11971142) and the Natural Science Foundation of Zhejiang Province (Grant no. LY19A010012).

References

[1] Abramowitz, M., Stegun, I.S., (1964), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U.S. Government Printing Office, Washington, 1046p.

[2] Alzer, H., Richards, K.C., (2020), A concavity property of the complete elliptic integral of the first kind, Integral Transf. Spec. Funct., 31(9), pp.758-768.

[3] Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M., (1997), Conformal Invariants, Inequalities, and Quasiconformal Maps, Wiley, New York, 505p.

[4] Borwein, J.M., Borwein, P.B., (1987), Pi and the AGM, Wiley, New York, 432p.

[5] Chen, Y.J., Wu, J.H., Zhao, T.H., (2023), On the absolute monotonicity of generalized elliptic integral of the first kind, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., 117(4), 143, pp.1-24.

[6] Chen, Y.J., Zhao, T.H., (2022), On the convexity and concavity of generalized complete elliptic integral of the first kind, Results in Math., 77(6), 215, pp.1-20.

[7] Chen, Y.J., Zhao, T.H., (2022), On the monotonicity for generalized elliptic integral of the first kind, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat., 116(2), 77, pp.1-21.

[8] Frasin, B.A., (2024), Starlikeness and convexity of integral operators involving Mittag-Leffler functions, TWMS J. Appl. Eng. Math., 14(3), pp.913-920.

[9] Heikkala, V., Vamanamurthy, M.K., Vuorinen, M., (2009), Generalized Elliptic integrals, Comput. Meth. Func. Theory., 9(1), pp.75-109.

[10] Jurkat, W.B., (1954), Questions of signs in power series, Proc. Amer. Math. Soc., 5(6), pp.964-970.

[11] Qiu, S.L., Vamanamurthy, M.K., Vuorinen, M., (1998), Some inequalities for the growth of elliptic integrals, SIAM J. Math. Anal., 29(5), pp.1224-1237.

[12] Rainville, E.D., (1960), Special Functions, Chelsea Publishing Company, New York, 390p.

[13] Richards, K.C., Smith, J.N., (2021), A concavity property of generalized complete elliptic integrals, Integral Transf. Spec. Funct., 32(3), pp.240-252.

[14] Tian, J.F., Yang, Z.H., (2022), Several absolutely monotonic functions related to the complete elliptic integral of the first kind, Results Math., 77(3), 109, pp.1-19.

[15] Tian, J.F., Yang, Z.H., (2023), Convexity and monotonicity involving the complete elliptic integral of the first kind, Results Math., 78(1), 29, pp.1-18.
[16] Wang, M.K., Chu, H.H., Li, Y.M., Chu, Y.M., (2020), Answers to three conjectures on the convexity of three functions involving complete elliptic integrals of the first kind, Appl. Anal. Discrete Math., 14(1), pp.255-271.
[17] Wang, M.K., Zhang, W., Chu, Y.M., (2019), Monotonicity, convexity and inequalities involving the generalized elliptic integrals, Acta Math. Sci. Ser. B (Engl. Ed.), 39(5), pp.1440-1450.
[18] Yang, Z.H., Qian, W.M., Chu, Y.M., (2018), Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 21(4), pp.1185-1199.
[19] Yang, Z.H., Tian, J.F., (2019), Convexity and monotonicity for elliptic integrals of the first kind and applications, Appl. Anal. Discrete Math., 13(1), pp.240-260.
[20] Yang, Z.H., Tian, J.F., (2022), Absolute monotonicity involving the complete elliptic integrals of the first kind with applications, Acta Math. Sci., 42B(3), pp.847-864.
[21] Zhao, T.H., Bhayo, B.A., Chu, Y.M., (2022), Inequalities for Generalized Grötzsch Ring Function, Comput. Methods Funct. Theory, 22(3), pp.559-574.
[22] Zhao, T.H., Wang, M.K., Chu, Y.M., (2021), Concavity and bounds involving generalized elliptic integral of the first kind, J. Math. Inequal., 15(2), pp.701-724.
[23] Zhao, T.H., Wang, M.K., Chu, Y.M., (2021), Monotonicity and convexity involving generalized elliptic integral of the first kind, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat., 115(2), 6, pp.1-13.

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